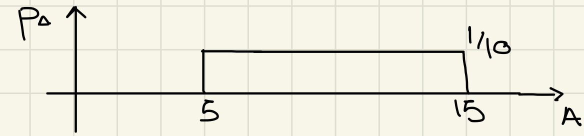
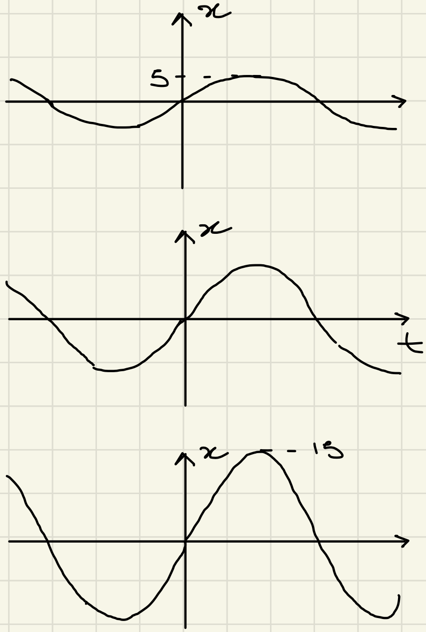
PROCESSI

Esercizi:

- $x(t) = A\sin(2\pi f_0 t)$ A v.c. $p(A) = \frac{1}{10}\mathrm{rect}\left(\frac{A}{10}-1\right)$





3 diverse realizzazioni di x

STAZIONARIETÀ IN SENSO LATO

$$E[x(t)] = \mu_x$$

$$E[x(t+\tau)\,x^*(t)] = R_x(\tau)$$

$$E[|x(t)|^2] = P_x$$

Devo ricavare $p_{x(t)}(x)$

↳ tempo t fissato

$$x(\bar{t}) = A\,\underbrace{\sin(2\pi f_0 \bar{t})}_{S}$$

$$x = A\,S \;\rightarrow\; A = \frac{x}{S}\,,\quad \frac{dx}{dA} = S$$

$$p_x(x) = \frac{p_A(x/S)}{\left|\frac{dx}{dA}\right|} = \frac{1}{S}\cdot\frac{1}{10}\,\mathrm{rect}\left(\frac{x}{10S}-1\right)$$

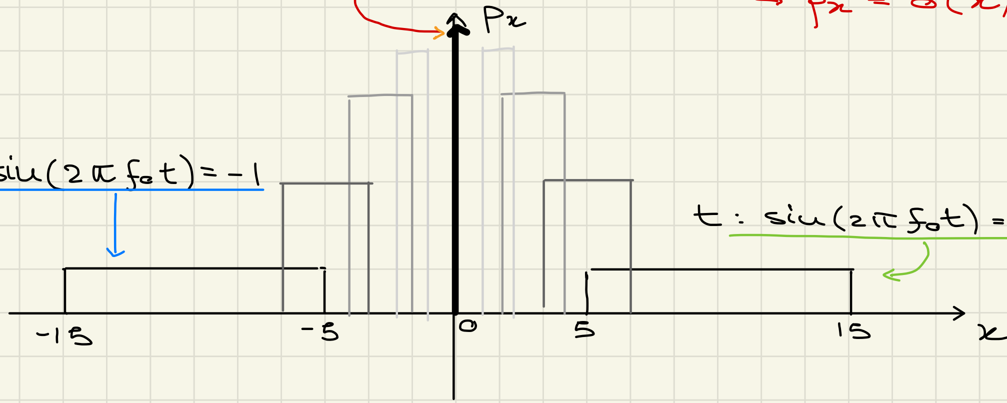
$$= \frac{1}{10\sin(2\pi f_0 t)}\,\mathrm{rect}\left(\frac{x-10\sin(2\pi f_0 t)}{10\sin(2\pi f_0 t)}\right)$$

per $t: \sin(2\pi f_0 t) = 0 \implies x = 0$ → non é più casuale!

↳ $p_x = \delta(x)$

$t: \sin(2\pi f_0 t) = -1$

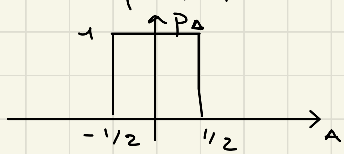
$t: \sin(2\pi f_0 t) = 1$



$$E[x(t)] = E[A \sin(2\pi f_0 t)] = E[A] \cdot \sin(2\pi f_0 t)$$
$$= 10 \cdot \sin(2\pi f_0 t)$$

$$\mathrm{var}[x(t)] = \mathrm{var}[A \sin(2\pi f_0 t)] = \mathrm{var}[A] \sin^2(2\pi f_0 t)$$
$$= \frac{10^2}{12} \sin^2(2\pi f_0 t)$$

- $x(t) = A + \cos(2\pi f_0 t)$ $\quad$ A v.c. $\quad p_A(A) = \mathrm{rect}(A)$

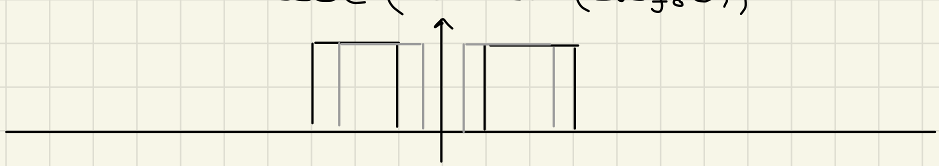


p_x ?

$x = A + c \longrightarrow A = x - c$

e $\frac{dx}{dA} = 1$

$$p_x(x) = \frac{p_A(x-c)}{\left|\frac{dx}{dA}\right|} =$$
$$= \mathrm{rect}(x - \cos(2\pi f_0 t))$$

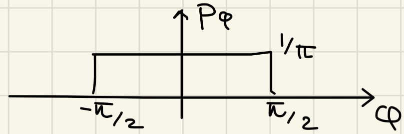


$$E[x(t)] = E[A] + \cos(2\pi f_0 t) = \cos(2\pi f_0 t)$$

$$\text{var}[x(t)] = \text{var}[A] = \frac{1}{12}$$

$\mu_x = \mu_x(t)$ quindi x NON è STAZIONARIO

- $x_n = \cos(2\pi f_0 n + \varphi) \quad \varphi$ v.c. $\quad p_\varphi(\varphi) = \frac{1}{\pi}\,\text{rect}\left(\frac{\varphi}{\pi}\right)$



$$E[x_n] = \int_{-\infty}^{+\infty} x_n(\varphi)\, p_\varphi(\varphi)\, d\varphi$$

$$= \int_{-\pi/2}^{\pi/2} \cos(2\pi f_0 n + \varphi)\frac{1}{\pi}\, d\varphi$$

$$= \frac{1}{\pi}\left[\sin\left(2\pi f_0 n + \frac{\pi}{2}\right) - \sin\left(2\pi f_0 n - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\cos(2\pi f_0 n) - (-\cos(2\pi f_0 n))\right]$$

$$= \frac{2}{\pi}\cos(2\pi f_0 n) \longrightarrow \text{NON è STAZIONARIO}$$

$$\text{var}[x_n] = E[x_n^2] - E^2[x] =$$

$$E[x_n^2] = \int_{-\pi/2}^{\pi/2} \cos^2(2\pi f_0 n + \varphi)\frac{d\varphi}{\pi} =$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\cos(4\pi f_0 n + 2\varphi) + 1}{2}\right) d\varphi$$

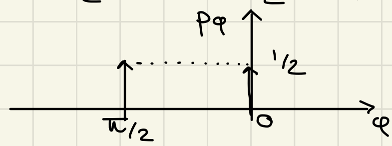
$$= \frac{1}{2\pi}\cdot\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1/2$$

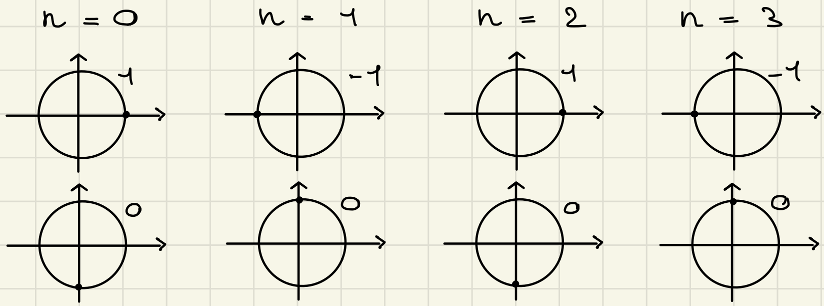
integrare la sinusoide sul suo periodo fa 0

$$\longrightarrow \text{var}[x_n] = \frac{1}{2} - \frac{4}{\pi^2}\cos^2(2\pi f_0 n)$$

- $x_n = \cos(\pi n + \varphi) \qquad p_\varphi(\varphi) = \frac{1}{2}\delta(\varphi) + \frac{1}{2}\delta(\varphi + \frac{\pi}{2})$

$$x_n = \begin{cases} \cos(\pi n) & p = 1/2 \\ \cos(\pi n - \frac{\pi}{2}) & p = 1/2 \end{cases}$$





$$\Longrightarrow \quad p_x(x_n) = \begin{cases} \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1) & n \text{ pari} \\ \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x+1) & n \text{ dispari} \end{cases}$$

$$E[x_n] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2} \quad n \text{ pari}$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot(-1) = -\frac{1}{2} \quad n \text{ dispari}$$

$$E[x_n^2] = \frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot 1^2 = \frac{1}{2} \quad n \text{ pari}$$

$$= \frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot(-1)^2 = \frac{1}{2} \quad n \text{ dispari}$$

$$\mathrm{var}[x_n] = \frac{1}{2} - \left(\pm\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

- $x(t) = 5\cos(2\pi f_0 t + \varphi)$

da 3π a 5π è come dire da $-\pi$ a π

$$p_\varphi(\varphi) = \frac{1}{2\pi}\,\mathrm{rect}\left(\frac{\varphi - 4\pi}{2\pi}\right) = \frac{1}{2\pi}\,\mathrm{rect}\left(\frac{\varphi}{2\pi}\right)$$

integro sul periodo

$$E[x(t)] = \int x(t)\, p_\varphi(\varphi)\, d\varphi = \int_{-\pi}^{\pi} 5\cos(2\pi f_0 t + \varphi)\, \frac{d\varphi}{2\pi}$$
$$= 0$$

$$\text{var}[x(t)] = E[x^2(t)] = \frac{25}{2\pi}\int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \varphi)\, d\varphi$$
$$= \frac{25}{4\pi}\int_{-\pi}^{\pi} \left(1 + \cos(4\pi f_0 t + 2\varphi)\right) d\varphi$$

integro 2 volte sul periodo

$$= \frac{25}{4\pi}\cdot 2\pi = \frac{25}{2}$$

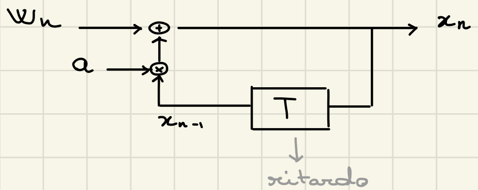
$$R_x(t_1, t_2) = E[\underbrace{x(t_1)\cdot x(t_2)}_{g(\varphi)}] = \int g(\varphi)\, p_\varphi(\varphi)\, d\varphi$$
$$= \frac{25}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi f_0 t_1 + \varphi)\cos(2\pi f_0 t_2 + \varphi)\, d\varphi$$
$$= \frac{25}{4\pi}\int_{-\pi}^{\pi} \left[\cos(2\pi f_0 (t_1 + t_2) + 2\varphi) + \cos(2\pi f_0 (t_1 - t_2))\right] d\varphi$$
$$= \frac{25}{4\pi}\cdot 2\pi \cos(2\pi f_0 (t_1 - t_2)) = \frac{25}{2}\cos(2\pi f_0 \underbrace{(t_1 - t_2)}_{\tau})$$

$$R_x = R_x(\tau)$$

$\Longrightarrow$ Il processo $x(t)$ è STAZIONARIO

- Processo autoregressivo

$$x_n = a\, x_{n-1} + w_n$$
$$E[w_n] = 0$$
$$E[w_n^2] = \sigma_w^2$$
$$E[w_n w_m] = 0, \quad n \neq m$$
$$R_w(n-m) = \sigma_w^2\, \delta_{n-m}$$



a t.c. x_n stazionario?

$$E[x_n] = E[a x_{n-1}] + E[w_n] = a\, E[x_{n-1}]$$

Se x_n é stazionario, il valore atteso non deve dipendere dal tempo, cioè $E[x_n] = E[x_{n-1}] = \mu_x$

$$\hookrightarrow \mu_x = a \mu_x \longrightarrow \begin{cases} a = 1, & \mu_x \neq 0 \\ a \text{ qualsiasi}, & \mu_x = 0 \end{cases}$$

- $\mu_x = 0$

$$var[x_n^2] = var[a^2 x_{n-1}^2 + w_n^2 + 2a x_{n-1} w_n]$$

↓ poiché x_n stazionario

$$\sigma_x^2 = a^2 \sigma_x^2 + \sigma_w^2 + 2a\, var[\cancel{x_{n-1} w_n}]$$

0, poiché w al campione n non può influenzare x al campione $n-1$, che é avvenuto all'istante prima

$$\sigma_x^2 = a^2 \sigma_x^2 + \sigma_w^2$$

$$\hookrightarrow a^2 = 1 - \frac{\sigma_w^2}{\sigma_x^2} \longrightarrow \begin{matrix} 0 \leq a^2 \leq 1 \\ 0 \leq |a| \leq 1 \end{matrix}$$

- $\mu_x \neq 0 \longrightarrow a = 1 \iff \sigma_w^2 = 0$ poiché $a^2 = 1 - \frac{\sigma_w^2}{\sigma_x^2}$

$$R_x(K) = E[x_{n+K}\, x_{\overset{n-m}{n}}] = E[(a x_{n-1+K} + w_{n+K}) x_n]$$
$$= a\, R_x(K-1) + E[\cancel{w_{n+K} x_n}]$$

0

$$R_x(K) = a\, R_x(K-1)$$

$$R_x(0) = \sigma_x^2, \quad R_x(1) = a \sigma_x^2 \quad R_x(2) = a^2 \sigma_x^2$$

$$\hookrightarrow R_x(K) = a^{|K|} \sigma_w^2$$

se x stazionario, $R_x(K) = R_x(-K)$

Processi Stazionari

$$R_x(t_1, t_2) = E[x(t_1)\, x^*(t_2)] = R_x(\underbrace{t_1 - t_2}_{\tau})$$

$$R_x(\tau) = E[x(t+\tau)\, x^*(t)] = R_x(t+\tau-t) = R_x(\tau)$$

$$R_x(0) = E[|x(t)|^2] = P_x \geq 0$$

$$R_x(-\tau) = E[x(t-\tau)\, x^*(t)] = \left(E[x(t)\, x^*(t-\tau)]\right)^*$$
$$= (R_x(t-t+\tau))^* = R_x^*(\tau)$$

$$|R_x(\tau)| \leq R_x(0)$$

$$E[x(t)] = 0 \implies R_x(\tau) = \mathrm{cov}(x(t+\tau), x(t))$$
$$\downarrow$$
$$u, v \in \mathbb{C}$$
$$\mathrm{cov}(u, v) = E[(u - E[u])(v - E[v])^*]$$

Cross-Correlazione: $R_{yx}(\tau) = E[y(t+\tau)\, x^*(t)]$

Es: $y(t) = a\, x(t - \tau_0)$

$$R_{yx}(\tau) = E[y(t+\tau)\, x^*(\tau)] = a\, E[x(t+\tau-\tau_0)\, x^*(t)]$$
$$= a\, R_x(\tau - \tau_0)$$

$$R_y(\tau) = E[y(t+\tau)\, y^*(\tau)] = |a|^2 E[x(t+\tau-\tau_0)\, x^*(t-\tau_0)]$$
$$= |a|^2 R_x(\tau)$$

$$y(t) = a\, x(t - \tau_a) + b\, x(t - \tau_b)$$

$$R_{yx}(\tau) = E[(a\, x(t+\tau-\tau_a) + b\, x(t+\tau-\tau_b)) \cdot x^*(t)] =$$
$$= a\, E[x(t+\tau-\tau_a)\, x^*(t)] + b\, E[x(t+\tau-\tau_b)\, x^*(t)] = a\, R_x(\tau-\tau_a) + b\, R_x(\tau-\tau_b)$$

$$R_y(\tau) = E\left[\left(a x(t+\tau-\tau_a) + b x(t+\tau-\tau_b)\right)\left(a x(t+\tau_a) + b x(t+\tau_b)\right)^*\right]$$
$$= |a|^2 R_x(\tau) + |b|^2 R_x(\tau) + a b^* R_x(\tau-(\tau_a-\tau_b)) + a^* b R_x(\tau-(\tau_b-\tau_a))$$

Stima Autocorrelazione

$$R_x(\tau) = E[x(t+\tau)\,x^*(t)]$$

$$\hat{R}_x(\tau) = \frac{1}{T_0}\int_{T_0} x(t+\tau)\,x^*(t)\,dt$$

$\hookrightarrow$ è una variabile casuale

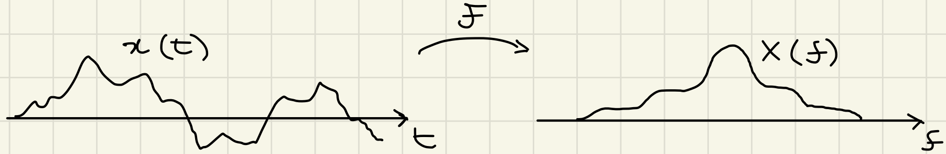
T_0 tempo di osservazione

$$E[\hat{R}_x(\tau)] = \frac{1}{T_0}\int_{T_0} \underbrace{E[x(t+\tau)\,x^*(t)]}\,dt =$$
$$= \frac{1}{T_0}\int_{T_0} R_x(\tau)\,dt = R_x(\tau)\,\frac{1}{T_0}\int_{T_0} dt =$$
$$= R_x(\tau)\,\frac{T_0-|\tau|}{T_0}$$

$\rightarrow$ fattore di polarizzazione

$$\mathcal{F}[\hat{R}_x(\tau)] = \frac{1}{T_0}|X_{T_0}(f)|^2$$

Caratterizzazione in Frequenza (di proc. staz.)



Valore atteso

X(f) non è staz.!

$$E[X(f)] = \int E[x(t)]\,e^{-j2\pi f t}\,dt = \int \mu_x e^{-j2\pi f t}\,dt = \mu_x \delta(f)$$

- Potenza

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \simeq \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi f n T} \Big|_T$$

somma di variabili incorrelate

$$E\left[|X(f)|^2\right] \longrightarrow \infty$$

- Densità spettrale di potenza

$$\left[S_x(f) = \lim_{T_0 \to \infty} \frac{1}{T_0} E\left[|X_{T_0}(f)|^2\right] \right] \qquad X_{T_0}(f) = \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi f t} dt$$

potenza per unità di frequenza

Teorema di Wiener

$$S_x(f) = \mathcal{F}\left[R_x(f)\right]$$

Dim.: $$\mathcal{F}^{-1}\left[S_x(f)\right] = \lim_{T_0 \to \infty} E\left[\frac{\mathcal{F}^{-1}\left[|X_{T_0}(f)|^2\right]}{T_0} \right] =$$

$$= \lim_{T_0 \to \infty} E\left[\hat{R}(\tau)\right] = \lim_{T_0 \to \infty} \frac{T_0 - |\tau|}{T_0} R_x(\tau) =$$

$$= R_x(\tau)$$

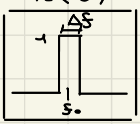
$$\boxed{\begin{aligned} S_x(f) &= \mathcal{F}\left[R_x(\tau)\right] = \int R_x(\tau) e^{-j2\pi f \tau} d\tau \\ R_x(\tau) &= \mathcal{F}^{-1}\left[S_x(f)\right] = \int S_x(f) e^{j2\pi f \tau} df \end{aligned}}$$

$$\boxed{P_x = R_x(0) = \int S_x(f)\, df}$$

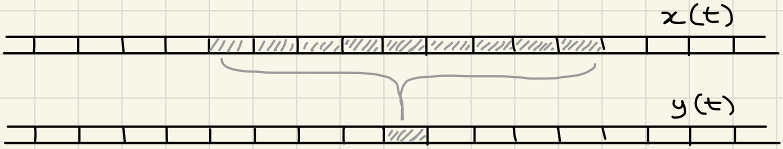
$P_x(f_0, \Delta f)$ potenza attorno alla frequenza f_0

$\dfrac{P_x(f_0, \Delta f)}{\Delta f}$ densità di potenza attorno a f_0

$h(t)$

$x(t) \longrightarrow$  $\longrightarrow y(t)$ $\qquad P_y = E\left[|y(t)|^2\right] = P_x(f_0, \Delta f)$

$y(t)$ è stazionario ? (x(t) sempre staz.)



$y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$ ogni valore di $y(t)$ è la somma pesata di tanti valori di x, ma poiché il peso è lo stesso per tutti (cioè $h(t)$) se prima istanti di tempo diversi erano equivalenti (stazionarietà) lo saranno ancora

$P_y = \int S_y(f)\, df$ per la stazionarietà di y

$$S_y(f) = \left\{ \lim_{T_0 \to 0} \frac{1}{T_0} E\left[|X(f)|^2\right] \right\} |H(f)|^2$$

$Y(f) = X(f)\, H(f)$

$|Y(f)|^2 = |X(f)|^2 |H(f)|^2$ $\quad\Big|\quad S_y(f) = S_x(f)\, |H(f)|^2$

$$P_x(f_0, \Delta f) = P_y = \int |H(f)|^2 S_x(f)\, df = \int_{\Delta f} S_x(f)\, df = S_x(f_0) \cdot \Delta f$$

$$\boxed{S_x(f) = \frac{P_x(f, \Delta f)}{\Delta f}}$$

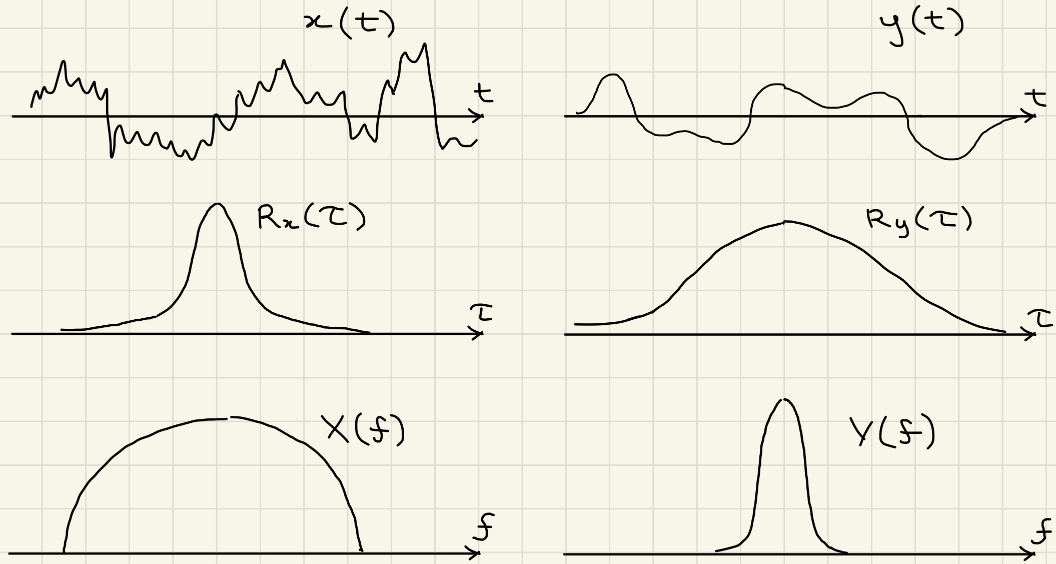
$$S_x(f) = \mathcal{F}[R_x(\tau)] \geq 0 \quad \underline{\text{sempre}}$$

$$R_x(\tau) = \mathcal{F}^{-1}[S_x(f)]$$

$\Downarrow$

simmetria hermitiana

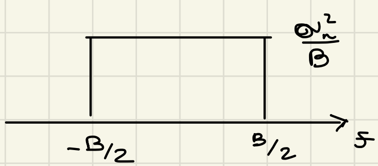
$$R_x(-\tau) = R_x^*(\tau)$$



Se $E[x(t)] = \mu_x \longrightarrow R_x(\tau) = C_x(\tau) + |\mu_x|^2$

$$S_x(f) = \mathcal{F}[C_x(\tau)] + |\mu_x|^2 \delta(f)$$

Rumore bianco $\quad n(t) \quad E[|n(t)|^2] = \sigma_n^2$



$$R_n(\tau) = \sigma_n^2 \, \frac{\sin(\pi 2B)}{\pi 2B}$$

Processi filtrati (LTI)

$x(t) \qquad y(t) = x(t) * h(t)$

$X(f) \qquad Y(f) = X(f) \cdot H(f)$

$S_x(f) \qquad S_y(f) = S_x(f)\,|H(f)|^2$

$R_x(\tau) \qquad R_y(\tau) = R_x(\tau) * \mathcal{F}^{-1}\left[|H(f)|^2\right] =$
$= R_x(\tau) * \underbrace{h(\tau) * h^*(-\tau)}_{R_h(\tau)}$

Densità Spettrale di Potenza

$$S_x(f) = \lim_{T_0 \to \infty} \frac{1}{T_0} E\left[|X_{T_0}(f)|^2\right] = \mathcal{F}[R_x(\tau)]$$

Cross - Spettro

$$S_{yx}(f) = \lim_{T_0 \to \infty} \frac{1}{T_0} E\left[Y_{T_0}(f) \cdot X^*_{T_0}(f)\right] = \mathcal{F}[R_{yx}(\tau)]$$

$$S_{yx}(f) = S_x(f) \cdot H(f) \qquad \left\{ \begin{aligned} S_{xy}(f) &= S_x(f)\,H^*(f) \\ R_{xy}(f) &= R_x(\tau) * h^*(-\tau) \end{aligned} \right\}$$

$$R_{yx}(\tau) = R_x(\tau) * h(\tau)$$

Esercizi:

- $y(t) = a\,x(t-\tau_0)$

  Dimostrare che $R_{yx}(\tau) = a\,R_x(\tau-\tau_0)$
  $R_y(\tau) = |a|^2 R_x(\tau)$.

  $h(t) = a\,\delta(t-\tau_0)$
  $y(t) = x(t) * h(t)$

$$H(f) = a\, e^{-j2\pi f \tau_0} \qquad |H(f)|^2 = |a|^2$$

$$S_{yx}(f) = S_x(f) \cdot a\, e^{-j2\pi f \tau_0}$$

$$S_y(f) = |a|^2 S_x(f)$$

$$R_{xy}(\tau) = R_x(\tau) * h(\tau) =$$
$$= R_x(\tau) * a\,\delta(\tau - \tau_0) = a R_x(\tau - \tau_0) \checkmark$$

$$R_y(\tau) = \mathcal{F}^{-1}\left[S_y(f)\right] = |a|^2 \mathcal{F}^{-1}\left[S_x(f)\right] = |a|^2 R_x(\tau) \checkmark$$

- $y(t) = a\,x(t - \tau_a) + b(t - \tau_b)$

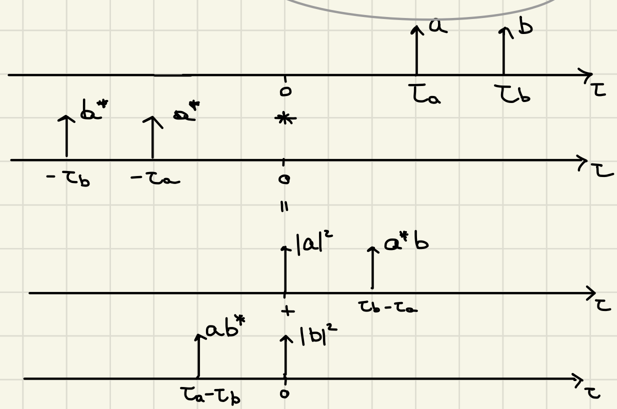
Dimostrare che

$$R_y(\tau) = a R_x(\tau - \tau_a) + b R_x(\tau - \tau_b)$$
$$R_{xy}(\tau) = (|a|^2 + |b|^2) R_x(\tau) + a b^* R_x(\tau - (\tau_a - \tau_b)) + a^* b R_x(\tau - (\tau_b - \tau_a))$$

$$h(t) = a\,\delta(t - \tau_a) + b\,\delta(t - \tau_b)$$

$$R_{yx}(\tau) = R_x(\tau) * h(\tau) = a R_x(\tau - \tau_a) + b R_x(\tau - \tau_b) \checkmark$$

$$R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau)$$

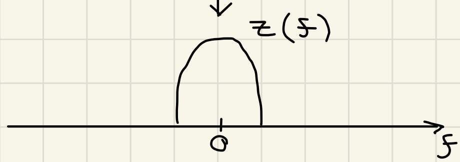


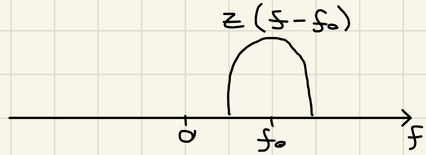
$$= R_x(\tau) * \left[(|a|^2 + |b|^2) + a b^* \delta(\tau - (\tau_a - \tau_b)) + a^* b\, \delta(\tau - (\tau_b - \tau_a)) \right] \checkmark$$

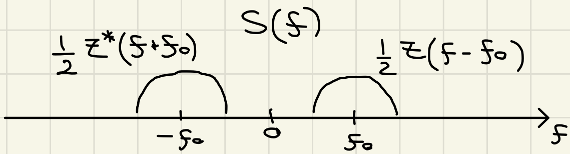
componente in fase   componente in quadratura

- $$s(t) = x(t)\cos(2\pi f_0 t) - y(t)\,\mathrm{sen}(2\pi f_0 t)$$
  $$= \mathrm{Re}\left[z(t)\, e^{+j2\pi f_0 t}\right]$$

$$z(t) = x(t) + j\,y(t)$$

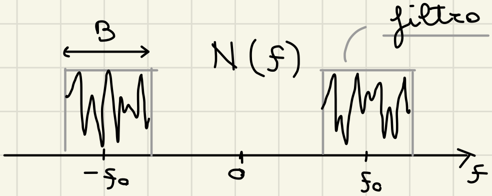


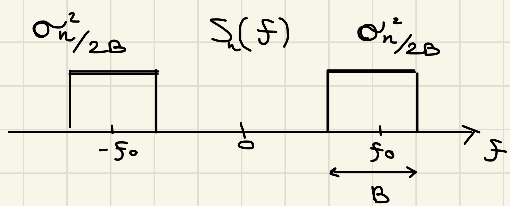




$$E_s = \frac{1}{2} E_z \qquad P_s = \frac{1}{2} P_z$$

$n(t)$ processo casuale (rumore o segnale)





$$\boxed{P_n = E\left[n^2(t)\right]} = BKT$$

$$n(t) = w_I(t)\cos(2\pi f_0 t) - w_Q(t)\,\mathrm{sen}(2\pi f_0 t)$$
$$= \mathrm{Re}\left[w(t)\, e^{+j2\pi f_0 t}\right]$$

$$w(t) = w_I(t) + j\,w_Q(t)$$

$$E[n(t)] = 0 \implies \overset{0}{E[w_I]}\cdot\cos - \overset{0}{E[w_Q]}\,\mathrm{sen} = 0$$

media nulla

$$E[n^2(t)] = \sigma_n^2 = \overbrace{E[w_I^2(t)]}^{\sigma_n^2}\cos^2(2\pi f_0 t) + \overbrace{E[w_Q^2(t)]}^{\sigma_n^2}\operatorname{sen}^2(2\pi f_0 t) - 2\underbrace{E[w_I(t)\,w_Q(t)]}_{\to 0}\operatorname{sen}(2\pi f_0 t)\cos(2\pi f_0 t)$$

$\Rightarrow E[w(t)] = 0$

$\Rightarrow E[|w(t)|^2] = 2\sigma_n^2$ $\qquad (E_z = 2E_s, \quad P_z = 2P_s)$

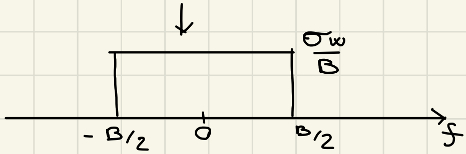
per avere $\sigma_w^2 = \sigma_n^2$
basterebbe porre $w(t) = \frac{1}{\sqrt{2}}\left(w_I(t) + j\,w_Q(t)\right)$

- $z_{Rx}(t) = g(t) + w(t)$

$$z_{Rx}^{\text{Filtr.}}(t) = z(t) * h(t) = g(t) * h(t) + w_F(t)$$

$w_F(t) \to w(t) * h(t)$

$$S_{w_F}(f) = S_w(f) \cdot |H(f)|^2$$

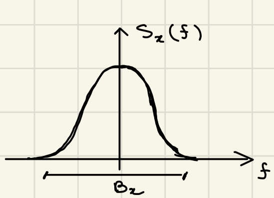


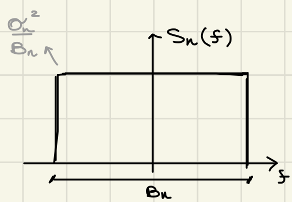
$$= \frac{\sigma_w^2}{B}|H(f)|^2$$

$$\sigma_{w_F}^2 = \int S_{w_F}(f)\,df = \frac{\sigma_w^2}{B}\int |H(f)|^2\,df = \frac{\sigma_w^2}{B}\int |h(t)|^2\,dt$$

- $x(t)$ processo stazionario
  $n(t)$ " "

$E[x(t)] = E[n(t)] = 0$
$\sigma_x^2 = E[|x(t)|^2] \qquad \sigma_n^2 = E$





$E[x(t+\tau)\,n^*(t)] = 0 \quad \forall \tau$

scorrelati

$B_n > B_x$

$$y(t) = \underbrace{x(t)}_{\text{segnale}} + \underbrace{n(t)}_{\text{rumore}}$$

a) $E[y(t)] = ?$

b) $P_y = E[|y(t)|^2] = ?$

c) $R_y(\tau) = ?$

d) $S_y(f) = ?$

e) $SNR = ?$

a) $E[y(t)] = E[x(t)] + E[n(t)] = 0$

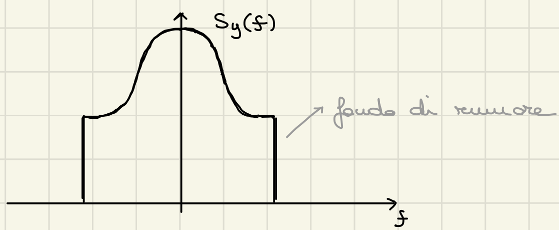
c) $R_y(\tau) = E[y(t+\tau)\, y^*(t)] = E[(x(t+\tau)+n(t+\tau))\cdot(x(t)+n(t))^*] =$

$$= E[\underbrace{x(t+\tau)x^*(t)}_{R_x(\tau)} + \underbrace{n(t+\tau)x^*(t)}_{R_{nx}(\tau)=0} + \underbrace{x(t+\tau)\cdot n^*(t)}_{R_{xn}(\tau)=0} + \underbrace{n(t+\tau)n^*(\tau)}_{R_n(\tau}] =$$

$$= \underbrace{R_x(\tau)}_{\mathcal{F}^{-1}[S_x(f)]} + \underbrace{R_y(\tau)}_{\sigma_n^2 \operatorname{sinc}(\tau B_n)}$$

b) $P_y = R_y(0) = R_x(0) + R_n(0) = \sigma_x^2 + \sigma_n^2$

d) $S_y(f) = \mathcal{F}[R_y(\tau)] = S_x(f) + S_n(f)$



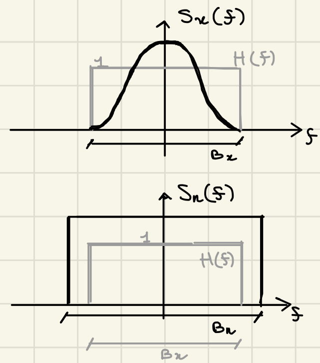
e) $SNR = \dfrac{P_x}{P_n} = \dfrac{\sigma_x^2}{\sigma_n^2}$

$y(t) \longrightarrow \boxed{h(t)} \longrightarrow v(t)$ con $h(t)$: $SNR_h > SNR$

$\Longrightarrow$ $h(t)$ deve essere un filtro con la banda di x (e amp. 1)

$H(f) = \operatorname{rect}\left(\dfrac{f}{B_x}\right) \longrightarrow h(t) = B_x \operatorname{sinc}(tB_x) = \dfrac{\sin(\pi t B_x)}{\pi t}$

$$v(t) = y(t) * h(t)$$
$$= \underbrace{x(t) * h(t)}_{x(t)} + \underbrace{n(t) * h(t)}_{w(t)}$$



$$SNR_h = \frac{\sigma_x^2}{\sigma_w^2}$$

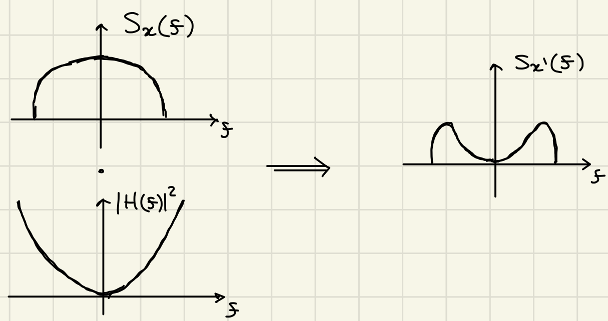
$$S_w(f) = |H(f)|^2 S_n(f)$$
$$P_w = \int_{-\infty}^{+\infty} S_w(f)\,df = \int_{-B_x/2}^{B_x/2} S_n(f) = \frac{\sigma_n^2}{B_n} B_x < \sigma_n^2$$

$$\Longrightarrow SNR_h = \frac{\sigma_x^2}{\sigma_n^2} \cdot \frac{B_n}{B_x}$$

- $x(t)$ processo stazionario con autocorrelazione $R_x(\tau)$

$$x'(t) = \frac{dx(t)}{dt} \qquad R_{x'x}(\tau) = ? \qquad S_{x'}(f) = ?$$

$$S_{x'x}(f) = H(f)\, S_x(f) = j2\pi f\, S_x(f)$$
$$R_{x'x}(\tau) = \mathcal{F}^{-1}[S_{x'x}(f)] = \frac{d}{d\tau} R_x(\tau)$$
$$S_{x'}(f) = |H(f)|^2 S_x(f) = 4\pi^2 f^2 S_x(f)$$ filtro passa-basso

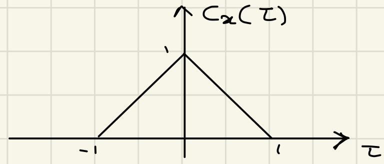


- $x(t)$ processo stazionario gaussiano reale

$$x(t) \sim \mathcal{N}(\mu_x, \sigma_x^2)$$

$P_x = 26$

$$C_x(\tau) = \begin{cases} 1-|\tau| & |\tau|<1 \\ 0 & |\tau|>1 \end{cases} \longrightarrow$$



$$\sigma_x^2 = C_x(0) = 1$$

$$P_x = \sigma_x^2 + \mu_x^2 \implies \mu_x = \pm 5$$

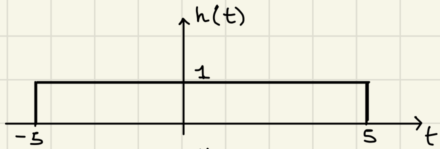
vanno bene entrambi, ne scelgo uno

$$x(t) \sim \mathcal{N}(5, 1)$$

$$R_x(\tau) = C_x(\tau) + \mu_x^2$$

$$S_x(f) = \mathcal{F}[C_x(\tau)] + 25\delta(f) = \left(\frac{\sin(\pi f)}{\pi f}\right)^2 + 25\,\delta(f)$$

$$h(t) = \mathrm{rect}\left(\frac{t}{10}\right)$$



$\downarrow$

$$H(f) = 10\,\mathrm{sinc}(10f)$$

$$y(t) = x(t) * h(t)$$

$$E[y(t)] = \int \overset{\overset{5}{\uparrow \text{staz.}}}{E[x(t-\tau)]} \cdot h(\tau)\,d\tau = 5 \underbrace{\int h(\tau)\,d\tau}_{10} = 50$$

$$\boxed{E[y(t)] = \mu_x \cdot H(0)}$$

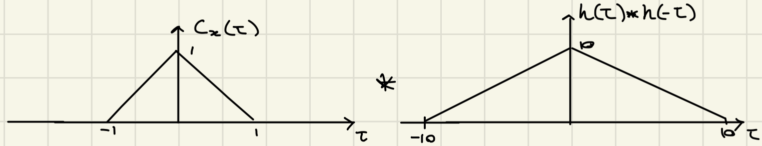
$$C_y(\tau) = R_y(\tau) - \mu_y^2$$

$$R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau) =$$

$$= \underbrace{C_x(\tau) * h(\tau) * h(-\tau)}_{C_y(\tau)} + \underbrace{\mu_x^2 * h(\tau) * h(-\tau)}_{\mu_y^2}$$

$$\Longrightarrow \quad C_y(\tau) = C_x(\tau) * \overbrace{h(\tau) * h(-\tau)}^{R_h(\tau)} =$$

$$=$$



Voglio sapere il <u>tempo di decorrelazione</u> cioè il τ oltre il quale la correlazione è sempre nulla (di C_y)

$$T_{c_y} = 2 + 20 = 22\,s$$

$$\longrightarrow t_{dec} = 11\,s$$

$$\sigma_y^2 = ?$$

$$\sigma_y^2 = C_y(0) \longrightarrow \quad C_y(\tau) = \int C_x(t) R_h(\tau - t)\, dt$$

$$C_y(0) = \int C_x(t) R_h(-t)\, dt$$

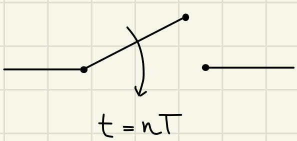
$$\rightarrow \quad \sigma_y^2 = \int_{-1}^{1} (1 - |t|) \cdot (10 - |t|)\, dt$$

$$= 2 \int_0^1 (10 - t - 10t + t^2)\, dt$$

$$= 2 \left[\frac{t^3}{3} - \frac{11}{2} t^2 + 10t \right]_0^1 = 2 \left[\frac{1}{3} - \frac{11}{2} + 10 \right]$$

$$= 2 \cdot \left[\frac{2 - 33 + 60}{6} \right] = \frac{29}{3}$$

Processi campionati

$x(t)$  x_n

$$R_{x_n}(k) = E[x_{n+k} \cdot x_n^*] = R_x(\tau = kT)$$

$$x_n = x(t = nT) \longrightarrow \quad R_{x_n}(k) = R_x(t = kT)$$

$$C_{x_n}(k) = C_x(t = kT)$$

$$y_n = x_n * h_n$$

$$R_{y_n}(k) = R_x(k) * h_k * h^*_{-k} \qquad R_{yx_n}(k) = R_x(k) * h_k$$

$$S_{x_n}(f) = \mathcal{F}\left[R_{x_n}(k)\right] = \frac{1}{T}\sum_{k=-\infty}^{+\infty} S_x\left(f-\frac{k}{T}\right)$$

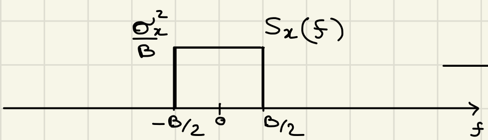
$$S_{y_n}(f) = |H(f)|^2 \, S_{x_n}(f) \qquad S_{yx_n}(f) = H(f)\, S_{x_n}(f)$$

$$P_{x_n} = T\int_{-1/2T}^{1/2T} S_{x_n}(f)\,df$$

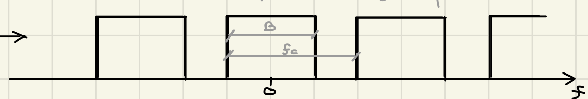
Esercizi:

- $x(t)$ processo $E[x(t)] = 0$, $E\left[|x(t)|^2\right] = \sigma_x^2$ bianco nella banda B.

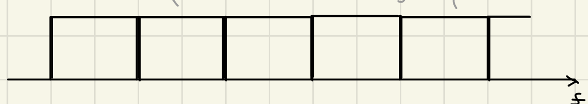
$$R_x(\tau) = \sigma_x^2 \frac{\sin(\pi B\tau)}{\pi B\tau} \longrightarrow R_{x_n}(k) = \sigma_x^2 \frac{\sin(\pi BkT)}{\pi BkT}$$



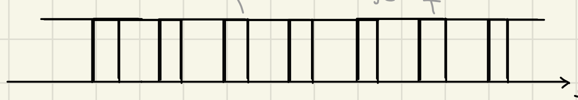
sovracampionato $f_c = \frac{1}{T} > B$



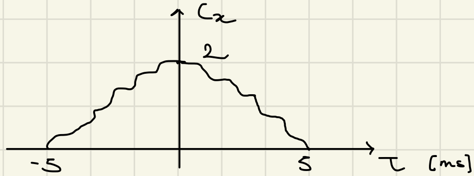
campionato critico $f_c = \frac{1}{T} = B$



sottocampionato $f_c = \frac{1}{T} < B$



- $x(t)$ processo $\mu_x = 1$, $\sigma_x^2 = 2$, $C_x(\tau) = 0 \ \forall |\tau| > 5ms$
  $x_n = x(t = nT)$ con $T = 17ms$



$S_{x_n}(f) = ?$

$R_x(\tau) = C_x(\tau) + |\mu_x|^2$

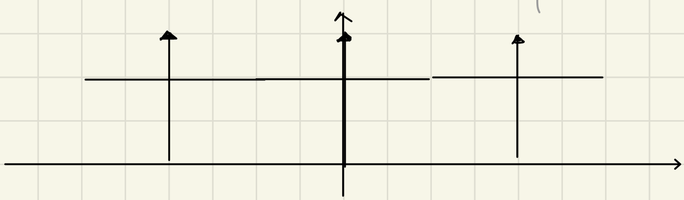
$P_x = \sigma_x^2 + |\mu_x|^2 = 3$

$C_{x_n}(k) = C_x(\tau = kT) = \sigma_x^2 \delta_k$
Poiché $T > 5ms$, $C_{x_n} \neq 0$ solo per $k = 0$.

$R_{x_n}(k) = \sigma_x^2 \delta_k + |\mu_x|^2$

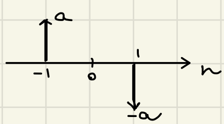
$$S_{x_k}(f) = \mathcal{F}[R_{x_n}(k)] = \sum_{k=-\infty}^{+\infty} R_{x_n}(k) e^{-j2\pi f kT}$$

$$= \underbrace{\mathcal{F}[\sigma_x^2 \delta_k]}_{\sigma_x^2} + \underbrace{\mathcal{F}[|\mu_x|^2]}_{\frac{|\mu_x|^2}{T}\delta(f)}$$



Progettare un filtro FIR (Finite Impulse Response) affinché il processo filtrato abbia $\mu_y = 0$ e $\sigma_y^2 = 16$

$y_n = x_n * h_n$, $\quad h_n = a(\delta_{n+1} - \delta_{n-1})$



$$E[y_n] = E\left[\sum_k x_{n-k} h_k\right] = \sum_k E[x_{n-k}] h_k = \mu_x \sum_k h_k = 0 = \mu_y$$

$$E[|y_n|^2] = E\Big[|a|^2 \underbrace{|x_{n+1}|^2}_{P_x} + |a|^2 \underbrace{|x_{n-1}|^2}_{P_x} - 2|a|^2 Re[x_{n+1} x_{n-1}^*]\Big]$$

$$\mathrm{Re}\left[E[x_{n+1} x_{n-1}^*]\right] \xRightarrow{\text{incorrelati}} \mathrm{Re}\left[\underbrace{E[x_{n+1}]}_{\mu_x} \underbrace{E[x_{n-1}]}_{\mu_x^*}\right]$$

** il processo x_n ha $C_{x_n} = 0 \ \forall n \neq 0$

$$\longrightarrow P_y = \left(|a|^2 \cdot 3 + |a|^2 \cdot 3 - 2 \cdot |a|^2 \cdot 1 \cdot 1\right) = |a|^2 \cdot 4 = \sigma_y^2 = 16$$

$$\Longrightarrow a = \pm 2 \qquad h_n = 2(\delta_{n+1} - \delta_{n-1})$$

Trasformate di Fourier in Matlab

$$X(f) = \int x(t) e^{-j2\pi f t} dt$$

$$X(f) = \underbrace{\left(\sum_n x_n e^{-j2\pi f nT}\right)}_{X_n(f)} \cdot \Delta t$$

tempo di campionamento

$$f_c = \frac{1}{\Delta t}$$

$$f \in \left(-\frac{f_c}{2}, \frac{f_c}{2}\right)$$

frequenza di campionamento

$$X = W_{kn} \cdot x_n \cdot \Delta t$$

$$W_{kn} = \begin{array}{cc} n & \\ \begin{bmatrix} \\ e^{-j2\pi f_k t_n} \\ \\ \end{bmatrix} & k \end{array} \qquad x_n = \begin{array}{cc} \begin{bmatrix} x_1 \\ \cdot \\ x_n \end{bmatrix} & n \end{array}$$

$$f = \left[-\frac{f_c}{2} \quad \cdot \quad \overset{k}{\underbrace{\cdot \quad \cdot}_{\Delta f}} \quad \frac{f_c}{2}\right]$$

$$x(t) = \int X(f) e^{j2\pi f t} df = \sum_k X_k e^{j2\pi f_k t_n} \cdot \Delta f \qquad \frac{1}{\Delta f} > T_0$$

$$N = \frac{T_0}{\Delta t} \qquad N_f = \frac{f_c}{\Delta f} = \frac{1}{\Delta t} \cdot \frac{1}{\Delta f} \longrightarrow N_f > N$$

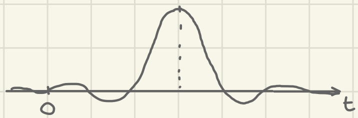
$$x_n = \sum_k X_k e^{j2\pi f_k t_n} \Delta f \cdot \Delta t$$

$$\longrightarrow \quad X_k = \sum_n x_n e^{-j2\pi f_k t_n} \iff X = W x$$

$$x_n = \sum_k X_k e^{j2\pi f_k t_n} \iff x = W^H X$$

inversa & complessa coniugata

$$X = \text{FFt}(x, N_f)$$



$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{kn}{N_f}}$$

$$f_k = \frac{k}{N_f} \qquad f_k = \left[0, \frac{1}{N_f}, \dots, \frac{N_f-1}{N_f}\right]$$

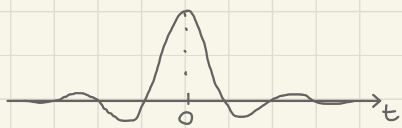
↳ mi fa vedere solo metà dell'asse delle frequenze

↓

FFt shift

$$X = \text{FFtshift}(\text{FFt}(x, N_f))$$

$$f_k = \left[-\frac{N-1}{2}, \dots, \frac{N-1}{2}\right]\frac{1}{N_f}$$

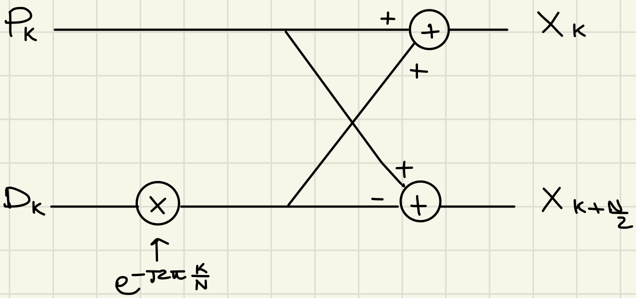


Su Matlab calcolare la trasformata di Fourier con le matrici è un'operazione molto gravosa. Per questo per calcolarla la funzione implementata in Matlab divide la sommatoria in 2 sommatorie di termini pari e dispari

$$X_k = \underbrace{\sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{kn}{N}}}_{(N)^2} = \underbrace{\sum_{n=0}^{N/2-1} x_{2n} e^{-j2\pi \frac{k \cdot 2n}{N}}}_{P_k \left(\frac{N}{2}\right)^2} + e^{-j2\pi \frac{k}{N}} \underbrace{\sum_{n=0}^{N/2-1} x_{2n+1} e^{-j2\pi \frac{k2n}{N}}}_{D_k \left(\frac{N}{2}\right)^2} \quad (N = N_f)$$

$$X_k = P_k + e^{-j2\pi \frac{k}{N}} D_k$$

$$X_{k+N/2} = P_k - e^{-j2\pi \frac{k}{N}} D_k$$



Se ripeto il processo e divido P_K e D_K posso raggiungere un numero di calcoli minimo pari a

$$\boxed{\frac{N}{2}\log_2 N}$$ → se N è potenza di 2

Stima

$\partial = \partial(\vartheta, \text{rumore}, \text{v.c.})$ ↓ parametri deterministici

$\hat{\vartheta}(\partial) = \hat{\vartheta}$ ↓ stimatore ↓ stima

$$\text{Err} = \hat{\vartheta} - \vartheta$$

Valutazione teorica:

$$E[\hat{\vartheta} - \vartheta] \begin{cases} \neq 0 & \text{polarizzato (biased)} \\ = 0 & \text{non polarizzato (unbiased)} \end{cases}$$

migliore

└ lo stimatore

$E[(\hat{\vartheta} - \vartheta)^2]$ Mean Root Square (MSR)

migliore se piccolo

Se unbiased:
MSR = var(E)

Valutazione sperimentale:

$$\begin{aligned} \partial_1 &= \hat{\vartheta}_1 \\ \partial_2 &= \hat{\vartheta}_2 \\ \partial_3 &= \hat{\vartheta}_3 \\ &\vdots \\ \partial_N &= \hat{\vartheta}_N \end{aligned}$$

$$\frac{1}{N} \sum_{n=1}^{N} (\hat{\vartheta}_n - \vartheta) \qquad \frac{1}{N} \sum_{n=1}^{N} (\hat{\vartheta}_n - \vartheta)^2$$

Media MSR

Es: $\partial_{1 \dots N} = \vartheta + w_n$ N campioni

autocorrelazione imp.

$w_n \sim \mathcal{N}(0, \sigma^2)$ $\quad E[w_n w_m] = 0$ se $n \neq m$

$\rightarrow \hat{\vartheta} = \partial_1 \qquad \varepsilon = \hat{\vartheta} - \vartheta = w_n$

$$E[\hat{\vartheta} - \vartheta] = E[\partial_1 - \vartheta] = E[w_n] = 0$$

$$E[(\hat{\vartheta} - \vartheta)^2] = E[w_n^2] = \sigma^2 \Longrightarrow \hat{\vartheta} = \vartheta \pm \sigma$$

$\rightarrow \hat{\vartheta} = \frac{1}{N} \sum_{n=1}^{N} \partial_n \qquad \varepsilon = \frac{1}{N} \sum_{n=1}^{N} (\vartheta + w_n) - \vartheta = \frac{1}{N} \sum_{n}^{N} w_n$

$$E[\hat{\vartheta} - \vartheta] = \frac{1}{N} \sum_{n=1}^{N} E[\partial_n] - \vartheta = 0$$

$$E[(\hat{\vartheta} - \vartheta)^2] = \frac{1}{N} \sigma^2 \Longrightarrow \hat{\vartheta} = \vartheta \pm \frac{\sigma}{\sqrt{N}}$$

$$E\left[\left(\frac{1}{N} \sum_{n=1}^{N} w_n\right)^2\right] = \frac{1}{N^2} E\left[\sum_{n=1}^{N} w_n^2\right] = \frac{1}{N^2} \sum_{n=1}^{N} E[w_n^2]$$

$$= \frac{1}{N^2} \cdot N \sigma^2 = \frac{1}{N} \sigma^2$$

Es: $x_n = \mu + w_n \qquad R_w(m) = \sigma_w^2 \delta_m$

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

$$\sqrt{E[\varepsilon^2]} = \frac{\sigma_w}{\sqrt{N}}$$ deviazione standard dell'err.

non posso + ricavare

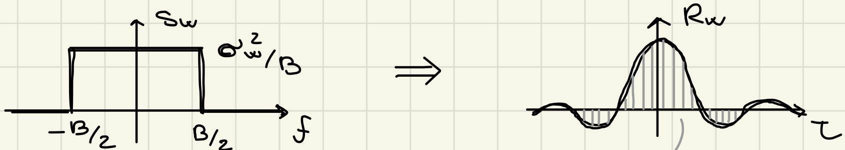
Se aumentassi N di molto (sovracampionam.) posso migliorare la stima? → Non per forza

xx l'autocorrelaz. del rum. non è + impulsiva

$$x(t) = \mu + w(t)$$

$$\hat{\mu} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt \qquad \varepsilon = \hat{\mu} - \mu \qquad E[\varepsilon] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} E[x(t)]\,dt$$

$E[w(t)] = 0$ bianco nella banda B



sovracamp. mi da un'autocorrelazione del rumore non più impulsiva

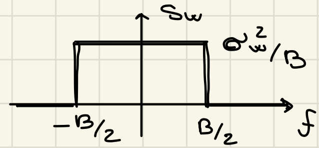
$$\varepsilon = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} w(t)\,dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}} w(\tau)\,\mathrm{rect}\left(\frac{\tau}{T_0}\right)d\tau$$

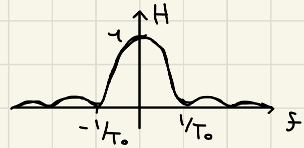
$$z(t) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} w(\tau)\,\mathrm{rect}\left(\frac{t-\tau}{T_0}\right)d\tau = w(t) * \frac{1}{T_0}\mathrm{rect}\left(\frac{t}{T_0}\right)$$

↳ $\varepsilon = z(0) \longrightarrow E[\varepsilon^2] = P_z = \int S_z(f)\,df$

$$S_z(f) = S_w(f)\,|H(f)|^2$$

$$\frac{\sigma_w^2}{B}\,\mathrm{rect}\left(\frac{f}{B}\right) \qquad \mathrm{sinc}^2(\pi f T_0)$$





$$\Longrightarrow \quad P_z = \frac{\sigma_w^2}{BT_0^2}\int_{-B/2}^{B/2}\left(\frac{\sin \pi f T_0}{\pi f}\right)^2 df$$

caso 1. $\quad T_0 \ll \frac{1}{B} \quad \Rightarrow \quad P_z \simeq \sigma_w^2$

tempo di decorrelazione del rumore

teo di Parceval

caso 2. $\quad T_0 \gg \frac{1}{B} \quad \Rightarrow \quad P_z \simeq \frac{\sigma_w^2}{BT_0^2}\int_{-\infty}^{+\infty}|h(t)|^2 dt = \frac{\sigma_w^2}{BT_0}$

$h(t) = \mathrm{rect}\left(\frac{t}{T_0}\right)$

BT_0 = numero equivalente di campioni indipendenti

$$\boxed{N = T_0 B}$$

$$x(t) \longrightarrow x_n = \mu + w_n$$

$T = \frac{1}{B}$ campionam. critico

$\downarrow$

autocorrelazione di w_n impulsiva

$$\boxed{E[\varepsilon^2] = \frac{\sigma_w^2}{N}}$$

Stima potenza

$z(t) \sim \mathcal{CN}(0, \sigma_z^2) \qquad z = x + jy$

$E[x^2] = E[y^2] = \frac{\sigma_z^2}{2} \qquad E[x] = E[y] = E[xy] = 0$

$E[zz^*] = \sigma_z^2$

$$* \; E[z_a z_b^* z_c^* z_d] = E[z_a z_b^*]\,E[z_d z_c^*] + E[z_a z_c^*]\,E[z_d z_b^*]$$

$$P_z = E[|z|^2] = \sigma_z^2$$

$\rightarrow \hat{P}_z = |z(t)|^2$ con t fissato $\qquad \varepsilon = \hat{P}_z - P_z = \hat{P}_z - \sigma_z^2$

$$E[\hat{P}_z] = E[|z(t)|^2] = \sigma_z^2$$

$E[\varepsilon] = E[P_z] - \sigma_z^2 = 0$ non polarizz

$$E[\varepsilon^2] = E[(\hat{P}_z - \sigma_z^2)^2] = E[\hat{P}_z^2] - \sigma_z^4$$

$$E[\hat{P}_z^2] = E[|z(t)|^4] = E[z z^* z z^*] =$$

$$* = \sigma_z^4 + \sigma_z^4 = 2\sigma_z^4$$

brutto

$\Rightarrow E[\varepsilon^2] = \sigma_z^4 \rightarrow$ deviazione standard della stima σ_z^2

$\rightarrow \hat{P}_z = \dfrac{1}{T_0}\displaystyle\int_0^{T_0} |z(t)|^2 \, dt$ o $\hat{P}_z = \dfrac{1}{N}\displaystyle\sum_n^N |z_n|^2$

$E[\hat{P}_z] = \sigma_z^2 \qquad E[\varepsilon^2] = \mathrm{var}[\hat{P}_z] = \dfrac{\sigma_z^2}{T_0 B}$

se $|z(t)|^2 = \sigma_z^2 + |w(t)|^2$

Riassunto: stima di media e stima di potenza

$$x(t) = \mu + w(t)$$

$$E[w(t)] = 0 \qquad E[|w(t)|^2] = \sigma_w^2$$

Tempo discreto. $\hat{\mu} = \frac{1}{N} \sum_n^N x_n$

Tempo continuo: $\hat{\mu} = \frac{1}{T_0} \int_{T_0} x(t)\,dt$

Errore di stima: $\varepsilon = \hat{\mu} - \mu$

$E[\varepsilon] = 0 \qquad E[|\varepsilon|^2] = \frac{\sigma_w^2}{N_{eq}}$

BT_0

$$z(t) \sim \mathbb{C}\mathcal{N}(0, \sigma_z^2) \qquad P_z = E[|z(t)|^2] = \sigma_z^2$$

$$|z(t)|^2 = P_z + \nu(t)$$

$$E[\nu(t)] = 0 \qquad \text{var}[\nu(t)] = \sigma_z^4$$

Tempo discreto. $\hat{P}_z = \frac{1}{N} \sum_n^N |z_n|^2$

Tempo continuo: $\hat{P}_z = \frac{1}{T_0} \int_{T_0} |z(t)|^2\,dt$

Errore di stima: $\varepsilon = \hat{P}_z - P_z$

$E[\varepsilon] = 0 \qquad E[|\varepsilon|^2] = \frac{\sigma_z^4}{N_{eq}}$

Stima autocorrelazione

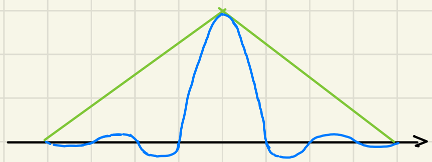
$x(t) \sim \mathbb{C}\mathcal{N}(0, \sigma_x^2)$

$\left[\hat{R}_x(\tau) = \frac{1}{T_0}\int_{T_0} x(t+\tau)x^*(t)\,dt\right]$ tempo continuo

$\left[\hat{R}_x(m) = \frac{1}{N}\sum_n^N x_{n+m}x_n^*\right]$ tempo discreto

Ricordiamo che:

$\left[E[\hat{R}_x(\tau)] = R_x(\tau)\,\frac{T_0 - |\tau|}{T_0}\right]$



Tanto più T_0 è piccolo $\longrightarrow$ tanto più il triangolo è stretto $\longrightarrow$ tanto peggiore è la stima

Consideriamo una nuova variabile casuale:

$\mu(t) = x(t+\tau)x^*(t)$
(oppure $\mu_n = \mu_{n+m}\mu_n^*$)

$\mu(t) = \underbrace{E[\mu(t)]}_{R_x(\tau)} + \nu(t)$

$E[\nu(t)] = 0$
$E[|\nu(t)|^2] = var[\mu(t)] = \sigma_\mu^2$

$var[\hat{R}_x(\tau)] = \frac{\sigma_\mu^2}{N_{eq}}$

$var[\mu(t)] = E[|\mu(t)|^2] - |\underbrace{E[\mu(t)]}_{}|^2 = \sigma_\mu^2$

$|R_x(\tau)|^2$

$E[|\mu(t)|^2] = E[x(t+\tau)x^*(t)x^*(t+\tau)x(t)] =$ $R_x(\tau)\cdot R_x^*(\tau)$
$= E[x(t+\tau)x^*(t)]\,E[x^*(t+\tau)x(t)] +$
$+ E[|x(t+\tau)|^2]\cdot E[|x(t)|^2] = |R_x(\tau)|^2 + \sigma_x^4$

$\sigma_x^2 \cdot \sigma_x^2$

$$\Longrightarrow \left[var[\mu(t)] = \sigma_x^4 + |R_x(\tau)|^2 - |R_x(\tau)|^2 = \sigma_x^4 \right]$$

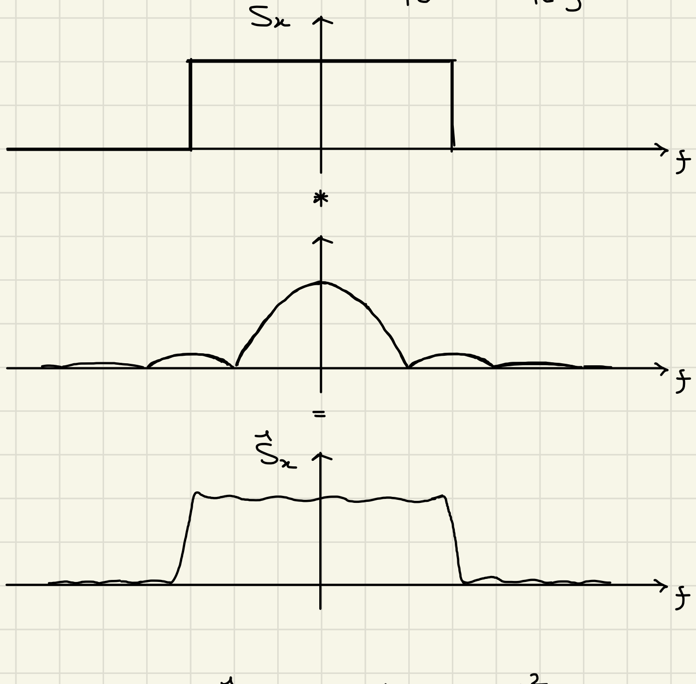
Stima densità spettrale di potenza

$$S_x(f) = \lim_{T_0 \to 0} \frac{1}{T_0} E\left[|X_{T_0}(f)|^2\right] = \mathcal{F}[R_x(\tau)]$$

$$\left[\hat{S}_x(f) = \frac{1}{T_0} E\left[|X_{T_0}(f)|^2\right] = \mathcal{F}[\hat{R}_x(\tau)]\right]$$ Periodogramma

$$E[\hat{S}_x(f)] = \mathcal{F}[E[\hat{R}_x(\tau)]] = \mathcal{F}\left[R_x(\tau)\, \frac{T_0 - |\tau|}{T_0}\right]$$

$$\hookrightarrow E[\hat{S}_x(f)] = S_x(f) * \frac{1}{T_0}\left(\frac{\sin \pi f T_0}{\pi f}\right)^2$$



$$var[\hat{S}_x(f)] \longrightarrow \quad \hat{S}_x(f) = \left|\frac{X_{T_0}(f)}{\sqrt{T_0}}\right|^2 = |A(f)|^2$$

$$E\left[|A(f)|^2\right] = E[\hat{S}_x(f)] = \sigma_A^2$$

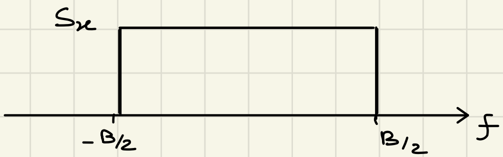
Se $A(f) \sim \mathcal{N}(0, \sigma_A^2)$ allora $var[|A(f)|] = \sigma_A^4$

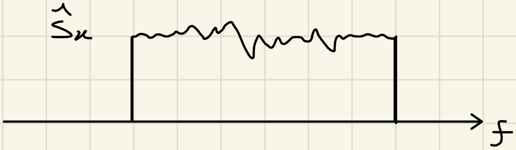
$$\Longrightarrow \left[var[\hat{S}_x(f)] = (E[S_x(f)])^2 \right]$$

Questo significa che: $\sigma_{\hat{S}_x} = E[\hat{S}_x(f)]$

Non sempre vale che $A(f) \sim \mathcal{CN}(0, \sigma_A^2)$

↳ allora la $var[\hat{S}_x(f)]$ é una stima

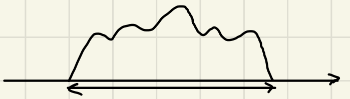




T_0 molto lungo

$E[\hat{S}_x(f)] = \frac{\sigma_x^2}{B}$ $\quad var[\hat{S}_x(f)] = \left(\frac{\sigma_x^2}{B}\right)^2$

$\hat{S}_x(f) = \frac{\sigma^2}{B} + \vartheta(f)$



$\hat{S}_0 = \frac{1}{B}\int_B \hat{S}_x(f)\,df$

$E[\hat{S}_0] = S_0 = \frac{\sigma_x^2}{B}$ $\quad var[\hat{S}_0] = \frac{S_0^2}{N_{eq}}$

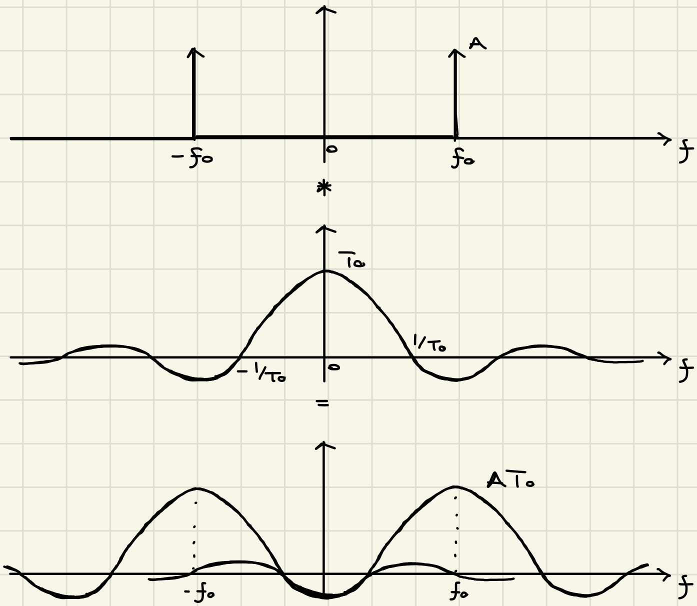
Distanza di correlazione $\Delta f = \frac{1}{T_0}$

$X_{T_0}(f) = \int_{T_0} x(t)e^{-j2\pi ft}dt$

$X_{T_0}(f) = X(f) * \frac{\sin(\pi f T_0)}{\pi f}$

⟶ tutte le trasformate che facciamo sono nella realtá convolute con un seno cardinale e scalate di un fattore $\times T_0$.

Per esempio una sinusoide:



$$E[X^*(f_1)X(f_2)] \begin{cases} = 0 & |f_1 - f_2| > \Delta f \\ \neq 0 & |f_1 - f_2| < \Delta f \end{cases}$$

distanza di correlazione

$$E[x^*(t_1)x(t_2)] \begin{cases} = 0 & |t_1 - t_2| > \tau_{dec} \\ \neq 0 & |t_1 - t_2| < \tau_{dec} \end{cases}$$

tempo di decorrelazione

Predizione Lineare

$x(t)$ processo $\quad E[x(t)] = 0, \quad R_x(\tau)$

Vogliamo stimare $x(t+\tau)$ noto $x(t)$

$\hat{x}(t+\tau)$ stima di $x(t+\tau)$

$[\hat{x}(t+\tau) = c \cdot x(t)]$ (dove $c = \text{const.}$)

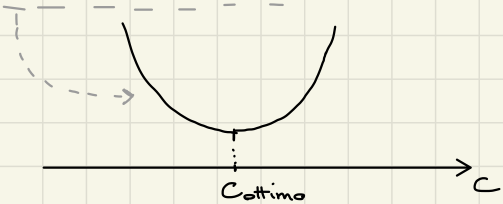
$[\varepsilon = \hat{x}(t+\tau) - x(t+\tau)]$

$[MSE = E[|\varepsilon|^2]]$ (Mean Square Error)

Scelgo c t.c. MSE sia minimo (mMSE)

$$\varepsilon = c \cdot x(t) - x(t+\tau)$$

$$MSE = E[|\varepsilon|^2] = E[x^2(t+\tau) - 2c\,x(t)x(t+\tau) + c^2 x^2(t)]$$
$$= R_x(0) - 2c R_x(\tau) + c^2 R_x(0)$$



Per trovare c_o
é sufficiente derivare:

$$\frac{d\,MSE}{dc} = 0 \Rightarrow \left[c_o = \frac{R_x(\tau)}{R_x(0)} \right]$$

Con questo valore c_o si minimizza la MSE

$$mMSE = R_x(0) - 2\frac{R_x^2(\tau)}{R_x(0)} + \frac{R_x^2(\tau)}{R_x(0)} = R_x(0)\left(1 - \frac{R_x^2(\tau)}{R_x^2(0)}\right)$$

Per processi a media nulla: $\dfrac{R_x(\tau)}{R_x(0)} = \rho_x(\tau)$

$$mMSE = R_x(0)\left(1 - \rho_x(\tau)\right)$$